A time-frequency density criterion for operator identification

Niklas Grip
Department of Mathematics, Luleå University of Technology
97187 Luleå, Sweden
niklas.grip@ltu.se

Götz E. Pfander
School of Engineering and Science, Jacobs University Bremen
28759 Bremen, Germany
g.pfander@jacobs-university.de

Peter Rashkov
Fachbereich Mathematik und Informatik, Philipps-Universität Marburg
35032 Marburg, Germany
rashkov@mathematik.uni-marburg.de

Abstract

We establish a necessary density criterion for the identifiability of time-frequency structured classes of Hilbert-Schmidt operators. The density condition is based on the density criterion for Gabor frames and Riesz bases in the space of square integrable functions. We complement our findings with examples of identifiable operator classes.

Key words and phrases : Operator identification, density criteria, Gabor frames, atomic Hilbert-Schmidt operator decompositions, sampling of operators.

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1 Introduction

The goal in operator identification is to recover an incompletely known operator from its action on a single input signal [20, 27]. In mathematical terms: for a normed linear space of operators $\mathcal{Z}$ mapping a set X into a normed linear space Y, we seek $g \in X$ such that the induced evaluation map

$$\Phi_g : \mathcal{Z} \to Y, \qquad H \mapsto Hg,$$

is bounded and boundedly invertible on its range. In short, we require that for some $g \in X$ there exist positive constants A and B with

$$A\|H\|_{\mathcal{Z}} \leq \|Hg\|_Y \leq B\|H\|_{\mathcal{Z}}, \quad H \in \mathcal{Z}.$$

This and similar inequalities will be represented by

$$\|Hg\|_Y \asymp \|H\|_{\mathcal{Z}}, \quad H \in \mathcal{Z},$$

from now on.

Identification of operators is important in numerous applications. For example, in mobile radio communications it is desirable to identify an unknown channel operator prior to information transmission. In radar applications, information on a target is gained through analyzing its response to a known sounding signal.

In this paper we focus on Hilbert-Schmidt operators on the space of square integrable functions on $\mathbb{R}$, $L^2(\mathbb{R})$. Hilbert-Schmidt operators on $L^2(\mathbb{R})$ are formally given by

$$Hf(x) = \iint \eta_H(t,\nu)\, e^{2\pi i \nu(x-t)}\, f(x-t)\, d\nu\, dt = \iint \eta_H(t,\nu)\, \pi_1(t,\nu) f(x)\, d\nu\, dt$$

with $\eta_H \in L^2(\mathbb{R}^2)$. The unitary time-frequency shift $\pi_d(\lambda)$ by $\lambda = (t,\nu) \in \mathbb{R}^{2d}$ is defined by

$$\pi_d(\lambda) f(x) = T_t M_\nu f(x) = e^{2\pi i \nu(x-t)}\, f(x-t), \quad f \in L^2(\mathbb{R}^d). \tag{1}$$

The space of Hilbert-Schmidt operators HS inherits the Hilbert space structure from $L^2(\mathbb{R}^2)$ by setting $\langle H, K\rangle_{HS} = \langle \eta_H, \eta_K\rangle_{L^2}$ and $\|H\|_{HS} = \|\eta_H\|_{L^2}$ [10, 12].

For $\lambda = (s,\omega; z, y) \in \mathbb{R}^4$ and H_0 Hilbert-Schmidt with spreading function η_0 we define the operator H_λ by

$$\eta_{H_\lambda} = \pi_2(\lambda)\eta_0 = T_{(s,\omega)} M_{(z,y)} \eta_0. \tag{2}$$

The central goal of this paper is to establish a density criterion on a not necessarily full-rank lattice Λ for the identifiability of the closed linear span of

$$(H_0, \Lambda) = \{H_\lambda\}_{\lambda \in \Lambda}, \tag{3}$$

that is, a necessary density condition on Λ for the existence of g with

$$\|Hg\|_{L^2(\mathbb{R})} \asymp \|H\|_{HS}, \quad H \in \overline{\operatorname{span}}(H_0, \Lambda). \tag{4}$$

The paper is structured as follows. Section 2 recalls general facts on modulation spaces, on Gabor Riesz bases and frames, and on Hilbert-Schmidt operators. In Section 3 we discuss the identification problem outlined above in detail and state our main result, Theorem 3.6. Theorem 3.6 is proven in Section 4. Section 5 provides some examples of identifiable classes of Hilbert-Schmidt operators. Also, the design of identifiers for the here considered operator families is discussed in Section 5.

2 Background

This section reviews some basic properties of Gabor Riesz sequences and frames in the Hilbert space of square integrable functions $L^2(\mathbb{R}^d)$ and in the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$.

A countable family of vectors $\{g_\lambda\}_{\lambda\in\Lambda}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* if

$$\|\{c_\lambda\}\|_{\ell^2(\Lambda)} \asymp \|\sum_{\lambda\in\Lambda} c_\lambda g_\lambda\|_{\mathcal{H}}, \quad \{c_\lambda\} \in \ell^2(\Lambda),$$

where $\ell^2(\Lambda)$ denotes the space of square summable sequences indexed by Λ.[1]

If only $\|\{c_\lambda\}\|_{\ell^2(\Lambda)} \geq A\|\sum_{\lambda\in\Lambda} c_\lambda g_\lambda\|_{\mathcal{H}}$, $\{c_\lambda\} \in \ell^2(\Lambda)$, for some positive A, then we refer to $\{g_\lambda\}_{\lambda\in\Lambda}$ as *Bessel sequence*. A *Riesz basis* is a Riesz sequence that spans $\mathcal{H}$.

A countable family $\{g_\lambda\}_{\lambda\in\Lambda}$ is a *frame* for $\mathcal{H}$ if

$$\|f\|_{L^2(\mathbb{R}^d)} \asymp \left\|\{\langle f, g_\lambda\rangle\}_{\lambda\in\Lambda}\right\|_{\ell^2(\Lambda)}, \quad f \in \mathcal{H}. \tag{5}$$

The set $\Lambda = M\mathbb{Z}^{2d} \subset \mathbb{R}^{2d}$ with M being a (not necessarily full rank) real $2d \times 2d$ matrix is called *lattice*. A Gabor system (g, Λ) in $L^2(\mathbb{R}^d)$ is the set of all time-frequency shifts (1) of the window function g by elements $\lambda = (x, \omega) \in \Lambda$, that is,

$$(g, \Lambda) = \{g_\lambda = \pi_d(\lambda)g : \lambda \in \Lambda\}.$$

The set (g, Λ) is called *Gabor Riesz sequence* if it is a Riesz sequence in $L^2(\mathbb{R}^d)$ and a *Gabor frame* if it is a frame for $L^2(\mathbb{R}^d)$.

Below, we shall utilize modulation space theory as developed by Feichtinger and Gröchenig [7, 8, 9, 14]. Let $\mathcal{S}(\mathbb{R})$ denote the space of Schwarz functions on $\mathbb{R}$ and $\mathcal{S}'(\mathbb{R})$ its dual of so-called tempered distributions. Let v_s, $s \in \mathbb{R}$, be the polynomial weight function $v_s(z) = (1 + |z|)^s$. The weighted mixed-norm sequence space $\ell^1_s(\Lambda)$ contains all sequences $\{c_\lambda\}_{\lambda\in\Lambda}$ with the property that

$$\|\{c_\lambda\}\|_{\ell^1_s(\Lambda)} = \sum_{\lambda\in\Lambda} |c_\lambda\, v_s(\lambda)| < \infty. \tag{6}$$

The weighted modulation space $M^1_s(\mathbb{R})$ consists of all those tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ with

$$\|f\|_{M^1_s} = \int |\langle f, \pi_1(z)\gamma\rangle\, v_s(z)|\, dz < \infty,$$

with $\gamma(x) = e^{-|x|^2}$, $x \in \mathbb{R}$, and where we refer to

$$V_\gamma f(z) = \langle f, \pi_1(z)\gamma\rangle, \quad z \in \mathbb{R}^2, \tag{7}$$

[1] Recall $\ell^2(\Lambda) = L^2(\Lambda, \nu)$ where ν is the counting measure on Λ.

as *short-time Fourier transform* of f with respect to the window γ.

The weighted modulation space $M_s^\infty(\mathbb{R})$ consists of all $f \in \mathcal{S}'(\mathbb{R})$ with norm

$$\|f\|_{M_s^\infty} = \sup_{z\in\mathbb{R}^2} |\langle f, \pi_1(z)\gamma\rangle v_s(z)| < \infty. \tag{8}$$

Note that the dual space of M_s^1 is M_{-s}^∞. If $s = 0$, we write simply M^1 and M^∞. For a detailed treatment of the theory of modulation spaces we refer to Chapters 11 and 12 of [14].

Hilbert-Schmidt operators on $L^2(\mathbb{R})$ are in one-to-one correspondence to their kernel [6, page 267], and, similarly, they can be represented by their time-varying impulse response h_H, their Kohn-Nirenberg symbol σ_H, and their spreading function η_H. In fact, formally,

$$\begin{aligned} Hf(x) &= \int \kappa_H(x,y) f(y)\, dy = \int h_H(t,x)\, f(x-t)\, dt \\ &= \iint \eta_H(t,\nu)\, e^{2\pi i\nu(x-t)}\, f(x-t)\, d\nu\, dt = \int \sigma_H(x,\xi) e^{2\pi i x\xi} \widehat{f}(\xi)\, d\xi. \end{aligned} \tag{9}$$

The functions $\kappa_H, h_H, \sigma_H, \eta_H$ are related by

$$\int \eta_H(t,\nu)\, e^{2\pi i\nu(x-t)}\, d\nu = h_H(t,x) = \kappa_H(x, x-t) = \int \sigma_H(x,\xi) e^{2\pi i\xi t}\, d\xi,$$

and

$$\|H\|_{HS} = \|\kappa_H\|_{L^2(\mathbb{R}^2)} = \|h_H\|_{L^2(\mathbb{R}^2)} = \|\eta_H\|_{L^2(\mathbb{R}^2)} = \|\sigma_H\|_{L^2(\mathbb{R}^2)}.$$

due to the unitarity of the L^2-Fourier transform $\mathcal{F}$ which is densely defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int f(x) e^{-2\pi i x\xi}\, dx.$$

For reference, we include the definition of Beurling density of $\mathbb{R}^d$. Let $\mathcal{B}_d(R)$ denote the ball in $\mathbb{R}^d$ centered at 0 with radius R and let $|\mathcal{M}|$ denote the cardinality of the set $\mathcal{M}$. The lower and upper Beurling densities of $\Lambda \subseteq \mathbb{R}^d$ are given by

$$\begin{aligned} D^-(\Lambda) &= \liminf_{R\to\infty} \inf_{z\in\mathbb{R}^d} \frac{|\Lambda \cap \{\mathcal{B}_d(R)+z\}|}{\operatorname{vol}\mathcal{B}_d(R)}, \\ D^+(\Lambda) &= \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d} \frac{|\Lambda \cap \{\mathcal{B}_d(R)+z\}|}{\operatorname{vol}\mathcal{B}_d(R)}. \end{aligned}$$

In case of $D^-(\Lambda) = D^+(\Lambda)$, we say that Λ has Beurling density $D(\Lambda) = D^-(\Lambda) = D^+(\Lambda)$. Note that the Beurling density of a lattice Λ equals the inverse of the Lebesgue measure of any measurable fundamental domain of Λ. See [21] for a more general concept of Beurling density.

3 Basic Observations and Main Result

If (H_0,Λ) in (3) is a Riesz sequence in the space of Hilbert-Schmidt operators, then

$$\overline{\operatorname{span}}(H_0,\Lambda)=\Big\{\sum_{\lambda\in\Lambda}c_\lambda\, H_\lambda\ :\quad \{c_\lambda\}\in\ell^2(\Lambda)\Big\}$$

and

$$\big\|\sum_{\lambda\in\Lambda}c_\lambda H_\lambda\big\|_{HS}\asymp\|\{c_\lambda\}\|_{\ell^2(\Lambda)},\quad \{c_\lambda\}\in\ell^2(\Lambda). \tag{10}$$

In this case, identifiability of $\overline{\operatorname{span}}(H_0,\Lambda)$ by g is equivalent to establishing

$$\big\|\sum_{\lambda\in\Lambda}c_\lambda H_\lambda g\big\|_2\asymp\|\{c_\lambda\}\|_{\ell^2(\Lambda)},\quad \{c_\lambda\}\in\ell^2(\Lambda), \tag{11}$$

that is, to showing that $\{H_\lambda g\}_{\lambda\in\Lambda}$ is a Riesz sequence in $L^2(\mathbb{R})$. Note that (10) corresponds to a Riesz sequence condition in the "large space" $L^2(\mathbb{R}^2)$, while (11) is a Riesz sequence condition on a similarly structured family of vectors in the "smaller space" $L^2(\mathbb{R})$. We shall generally assume that the weaker condition, (H_0,Λ) is a Riesz sequence, holds and focus on the question whether the set $\{H_\lambda g\}_{\lambda\in\Lambda}$ is a Riesz sequence for some g.

Remark 3.1 (*i*) We established that (10) and (11) implies (4). Moreover, if g is square integrable, then

$$\|\sum_{\lambda\in\Lambda}c_\lambda H_\lambda g\|_{L^2}\le\|\sum_{\lambda\in\Lambda}c_\lambda H_\lambda\|_{HS}\,\|g\|_{L^2},\quad \{c_\lambda\}\in\ell^2(\Lambda),$$

and (10) can be replaced with the condition that (H_0,Λ) is a Bessel sequence to obtain (4). This argument is not always applicable as we shall generally allow g to be a tempered distribution and use the fact that some spaces of operators map spaces of distributions into $L^2(\mathbb{R})$. For example, operators in so-called operator Paley-Wiener space

$$OPW^2(S)=\{H\in HS(L^2(\mathbb{R})):\ \operatorname{supp}\eta_H\subseteq S\}$$

map boundedly the modulation space $M^\infty(\mathbb{R})$ defined in (8) to $L^2(\mathbb{R})$ in case that S is compact [23].
(*ii*) The identifiability of $\overline{\operatorname{span}}(H_0,\Lambda)$ neither implies (10) nor (11). Indeed, in some cases $\overline{\operatorname{span}}(H_0,\Lambda)$ is identifiable, $\overline{\operatorname{span}}(H_0,\Lambda)=\overline{\operatorname{span}}(H_0,2\Lambda)$, and $(H_0,2\Lambda)$ and $\{H_\lambda g\}_{\lambda\in 2\Lambda}$ are Riesz sequences while (H_0,Λ) and $\{H_\lambda g\}_{\lambda\in\Lambda}$ are not.

The framework developed here is motivated in part by the following well known results from time-frequency analysis. The first result gives a necessary condition on a Gabor system to form a Riesz sequence (see [17] and references therein).

Theorem 3.2 *Let* $\Gamma = \left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)\mathbb{Z}^2$ *be a full rank lattice in* $\mathbb{R}^2$*. If there exists* $g \in L^2(\mathbb{R})$ *such that* $(g,\Gamma) = \{\pi_1(\gamma)g\}_{\gamma\in\Gamma}$ *is a Riesz sequence in* $L^2(\mathbb{R})$*, then the Beurling density* $D(\Gamma) = \left|\det\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)\right|^{-1}$ *of* Γ *satisfies* $D(\Gamma) \leq 1$.

Theorem 3.2 is a special case of the herein established framework of operator identification. Indeed, set

$$\Lambda_\Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}^T \Gamma = \begin{pmatrix} a_1 & b_1 & b_1 & 0 \\ a_2 & b_2 & b_2 & 0 \end{pmatrix}^T \mathbb{Z}^2$$

and observe that Lemma 4.1 implies that for any Hilbert-Schmidt operator H_0 we have

$$\overline{\operatorname{span}}(H_0, \Lambda_\Gamma) = \overline{\operatorname{span}}\{\pi_1(\gamma) \circ H_0\}_{\gamma\in\Gamma}.$$

So g identifies $\overline{\operatorname{span}}(H_0,\Lambda_\Gamma)$ if and only if $\{\pi_1(\gamma)(H_0 g)\}_{\gamma\in\Gamma}$ is a Riesz sequence. It follows that $D(\Gamma) \leq 1$ is necessary for the identifiability of $\overline{\operatorname{span}}(H_0,\Lambda_\Gamma)$.[2] The condition $D(\Gamma) \leq 1$ implies that Λ_Γ is not too dense in the two-dimensional "tilted" plane $\mathbb{R} \times \{y = z\} \times \{0\} \subseteq \mathbb{R}^4$.

The second result motivating this paper plays a critical role in the analysis of slowly time-varying operators [18, 4] and in the recently developed sampling theory for operators [20, 27, 24].

Theorem 3.3 *Let* $\mathcal{M} = \left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)\mathbb{Z}^2 \subseteq \mathbb{R}^2$ *be a full-rank lattice and* $H_0 : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$*,* $H_0 f = \rho\cdot(f * r)$*, be a product-convolution Hilbert-Schmidt operator with* $\rho, \widehat{r}$ *smooth and compactly supported. If there exists a tempered distribution* g *such that* $\{\pi_1(\gamma)H_0\pi_1(\gamma)^* g\}_{\gamma\in\mathcal{M}}$ *is a Riesz sequence in* $L^2(\mathbb{R})$*, then* $D(\mathcal{M}) \leq 1$.

Similarly to above, Lemma 4.1 implies that setting

$$\Lambda_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \mathcal{M} = \begin{pmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_1 \end{pmatrix}^T \mathbb{Z}^2$$

indicates that Theorem 3.3 may be considered a special case of our framework.

Theorems 3.2 and 3.3 motivate the central question in this paper which we paraphrase as follows.

Question 3.4 Can we define a density $\widetilde{D}$ on lattices $\Lambda = M\mathbb{Z}^2 \subseteq \mathbb{R}^4$ so that for a positive constant C we have $\widetilde{D}(\Lambda) > C$ implies $\overline{\operatorname{span}}(H_0,\Lambda)$ is not identifiable whenever (H_0,Λ) is a Riesz sequence?

We choose the following "Beurling-type" density for sets of points Λ lying within general two-dimensional subspaces of $\mathbb{R}^4$.

[2] Our reasoning uses $Hg \in L^2(\mathbb{R})$ and not $g \in L^2(\mathbb{R})$. Indeed, the flexibility of choosing not square integrable g is quite beneficial. In fact, for some (H_0,Λ) we must choose $g \in M^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$ to achieve that $\{H_\lambda g\}_{\lambda\in\Lambda}$ is a Riesz sequence in $L^2(\mathbb{R})$.

Definition 3.5 *The "two-dimensional" upper and lower Beurling densities (or for short 2-Beurling density) of* $\Lambda \subseteq \mathbb{R}^4$ *are given by*

$$
\begin{aligned}
D^-_{(2)}(\Lambda) &= \sup_{U \in \mathcal{U}} \liminf_{R\to\infty} \inf_{z\in U} \frac{|\Lambda \cap \{\mathcal{B}_4(R)+z\}|}{\pi R^2},\\
D^+_{(2)}(\Lambda) &= \sup_{U \in \mathcal{U}} \limsup_{R\to\infty} \sup_{z\in U} \frac{|\Lambda \cap \{\mathcal{B}_4(R)+z\}|}{\pi R^2},
\end{aligned} \tag{12}
$$

where $\mathcal{U}$ *denotes the set of two-dimensional affine subspaces of* $\mathbb{R}^4$ *and* $\mathcal{B}_4(R)$ *is the centered open ball with radius* R *in* $\mathbb{R}^4$.

If $D^+_{(2)}(\Lambda) = D^-_{(2)}(\Lambda)$, *then* Λ *has uniform 2-Beurling density* $D_{(2)}(\Lambda) = D^-_{(2)}(\Lambda)$.

Observe that with

$$\Lambda = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}^T \mathbb{Z}^2$$

we have

$$
\begin{aligned}
D_{(2)}(\Lambda) =&\big[(a_1b_2-a_2b_1)^2+(a_1c_2-a_2c_1)^2\\
&+(a_1d_2-a_2d_1)^2+(b_1c_2-b_2c_1)^2\\
&+(b_1d_2-b_2d_1)^2+(c_1d_2-c_2d_1)^2\big]^{-1/2}.
\end{aligned} \tag{13}
$$

Hence, for $\Lambda_{\mathcal{M}} = \left(\begin{smallmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{smallmatrix}\right)^T \mathbb{Z}^2$ in Theorem 3.3 we have $D_{(2)}(\Lambda_{\mathcal{M}}) = |a_1b_2 - a_2b_1|^{-1} = \big|\det\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)\big|^{-1}$, and for $\Lambda_\Gamma = \left(\begin{smallmatrix} a_1 & b_1 & b_1 & 0 \\ a_2 & b_2 & b_2 & 0 \end{smallmatrix}\right)^T \mathbb{Z}^2$ in Theorem 3.2 we have $D_{(2)}(\Lambda_\Gamma) = 2^{-1/2}\big|\det\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)\big|^{-1}$.

The main result in this paper provides a necessary condition on Λ for the existence of g so that $\{H_\lambda g\}_{\lambda\in\Lambda}$ is a Riesz sequence. M^1_s and M^∞ denote modulation spaces whose definitions we recalled in Section 2.

Theorem 3.6 *Let* $\Lambda = M\mathbb{Z}^2 \subseteq \mathbb{R}^4$ *and* H_0 *be an operator with* $\eta_{H_0} \in M^1_s(\mathbb{R})$, $s > 2$. *If* (H_0, Λ) *is Riesz and* $\overline{\text{span}}(H_0,\Lambda)$ *is identifiable by* $g \in M^\infty(\mathbb{R})$ *then* $D_{(2)}(\Lambda) \leq \sqrt{2}$.

If the first or the last row of M is zero, then the bound on $D_{(2)}(\Lambda)$ can be improved. Identifiability then implies $D_{(2)}(\Lambda) \leq 1$. This is the case in Theorem 3.3 and Theorem 3.2 respectively (in Theorem 3.2 even $D_{(2)}(\Lambda) \leq 2^{-1/2}$). It is not clear whether the constant $\sqrt{2}$ in Theorem 3.6 can be improved in the general case.

Identifiability of $\overline{\text{span}}(H_0,\Lambda)$ depends on both, H_0 and the lattice Λ, so it is not surprising that there exists no $c > 0$ with $D^+_{(2)}(\Lambda) < c$ implies $\overline{\text{span}}(H_0,\Lambda)$ is identifiable. In fact, it is easy to construct for any $\epsilon > 0$ a Riesz sequence (H_0,Λ) with $D_{(2)}(\Lambda) \leq \epsilon$ but $\overline{\text{span}}(H_0,\Lambda)$ is not identifiable (see Example 5.4 and [13]).

We would like to emphasize that the operator outputs considered herein are in the "small" space $L^2(\mathbb{R})$ while the kernels and spreading functions of the operators are in the "larger" space of $L^2(\mathbb{R}^2)$ functions. This dimension mismatch implies that a single evaluation map $H \mapsto Hg$ cannot identify the space of Hilbert-Schmidt operators as a whole, just as a degree of freedom counting argument shows that the space of complex $n \times n$ matrices requires the use of n identifiers for identification.

4 Proof of Theorem 3.6

The following lemmas are used in the proof of Theorem 3.6.

Lemma 4.1 *Let η_0 denote the spreading function of $H_0 \in HS(\mathbb{R})$. Then the operator $T_a M_b T_{-c} H_0 T_c M_d$ has spreading function*

$$\eta_{T_a M_b T_{-c} H_0 T_c M_d} = T_{a,b+d} M_{b,c} \eta_0, \quad a, b, c, d \in \mathbb{R}.$$

Hence,

$$\eta_H = \sum_{\lambda=(s,\omega,\xi,y)\in\Lambda} c_\lambda \eta_{H_\lambda} = \sum_{\lambda=(s,\omega,\xi,y)\in\Lambda} c_\lambda \pi_2(\lambda)\eta_0.$$

converges in L^2-norm if and only if

$$H = \sum_{\lambda=(s,\omega,\xi,y)\in\Lambda} c_\lambda T_s M_\xi T_{-y} H_0 T_y M_{\omega-\xi}$$

converges in $HS(\mathbb{R})$.

Proof. A change of variables $t = t - a$, $\nu = \nu - b - d$ and the relation $T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x$ implies

$$\begin{aligned}
&\iint T_{a,b+d} M_{b,c} \eta_0(t,\nu) f(x-t) e^{2\pi i \nu(x-t)}\, d\nu\, dt = \\
&\quad = \iint e^{2\pi i(b(t-a)+c(\nu-b-d))} \eta_0(t-a, \nu-b-d) f(x-t) e^{2\pi i\nu(x-t)}\, d\nu\, dt \\
&\quad = \iint e^{2\pi i(bt+c\nu)} \eta_0(t,\nu) f(x-t-a) e^{2\pi i(\nu+b+d)(x-t-a)}\, d\nu\, dt \\
&\quad = \iint e^{2\pi i(bt+c\nu)} \eta_0(t,\nu) T_{t+a} M_{\nu+b+d} f(x)\, d\nu\, dt \\
&\quad = T_a M_b T_{-c} \iint \eta_0(t,\nu) T_t M_\nu T_c M_d f(x)\, d\nu\, dt \\
&\quad = T_a M_b T_{-c} H_0 T_c M_d f(x).
\end{aligned}$$

Hence, in particular,

$$\begin{aligned}(Hf)(x) &= \iint \eta_H(t,\nu) f(x-t) e^{2\pi i\nu(x-t)}\, d\nu\, dt \\ &= \sum_{\lambda=(s,\omega,\xi,y)\in\Lambda} c_\lambda \iint T_{s,\omega} M_{\xi,y} \eta_0(t,\nu) f(x-t) e^{2\pi i\nu(x-t)}\, d\nu\, dt \\ &= \sum_{\lambda=(s,\omega,\xi,y)\in\Lambda} c_\lambda T_s M_\xi T_{-y} H_0 T_y M_{\omega-\xi} f(x).\end{aligned}$$

□

Lemma 4.2 *Let $p, q \in C_c^\infty(\mathbb{R}^d)$ and let P be the product-convolution operator with spreading function $\eta_P = p \otimes q$. Then there exist functions $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$|Pf(x)| \le \|f\|_{M^\infty(\mathbb{R}^d)} |\psi_1(x)|, \quad |\mathcal{F}Pf(\omega)| \le \|f\|_{M^\infty(\mathbb{R}^d)} |\psi_2(\omega)|, \quad f \in M^\infty(\mathbb{R}^d).$$

Lemma 4.2 is a straightforward generalization of Lemma 3.4 in [20] and its proof is omitted. Lemma 4.2 with $d = 1$ is used to prove the following result.

Lemma 4.3 *For H with spreading function $\eta_0 \in M_s^1(\mathbb{R})$, $s > 2$, there exist $\varphi_1(t), \varphi_2(t) = O(t^{-s})$ with*

$$|Hf(x)| \le \|f\|_{M^\infty(\mathbb{R})} \varphi_1(x), \quad |\mathcal{F}Hf(\omega)| \le \|f\|_{M^\infty(\mathbb{R})} \varphi_2(\omega), \quad f \in M^\infty(\mathbb{R}).$$

The decay estimates show that $Hg \in L^2(\mathbb{R})$ [11, (2.52)].

Proof. By choosing parameters a, b, c, d with $ac < 1, bd < 1$ and functions $p, q \in C_c^\infty(\mathbb{R})$ with $[-\frac{a}{2}, \frac{a}{2}] \subset \operatorname{supp} p \subset [-\frac{1}{2c}, \frac{1}{2c}], [-\frac{b}{2}, \frac{b}{2}] \subset \operatorname{supp} q \subset [-\frac{1}{2d}, \frac{1}{2d}]$ we obtain a Gabor frame $(\eta_P, a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z})$ for $L^2(\mathbb{R}^2)$, where $\eta_P = p \otimes q$ (see [25, pages 22-23] or [26]). Because $\eta_P \in \mathcal{S}(\mathbb{R}^2) \subset M_s^1(\mathbb{R}^2)$, the Gabor system $(\eta_P, a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z} \times d\mathbb{Z})$ is a universal Banach frame according to the definition of Gröchenig for all modulation spaces $M_s^1(\mathbb{R}^2)$ [14, Section 13.6]. The main result from [16] states that the canonical dual window satisfies $\tilde{\eta}_P \in M_s^1(\mathbb{R}^2)$. Let us denote by $\ell_s^1(\mathbb{Z}^4)$ the weighted mixed-norm sequence space containing all sequences $\{c_z\}_{z\in\mathbb{Z}^4}$ such that the norm

$$\|\{c_z\}\|_{\ell_s^1(\mathbb{Z}^4)} = \sum_{z\in\mathbb{Z}^4} |c_z\, v_s(z)| < \infty,$$

where v_s is defined by (6).

Then the result from [14, Corollary 12.2.6] shows that the series expansion

$$f = \sum_{k,l,m,n\in\mathbb{Z}} \langle f, T_{ak,bl} M_{cm,dn} \tilde{\eta}_P \rangle T_{ak,bl} M_{cm,dn} \eta_P, \quad f \in M_s^1(\mathbb{R}^2) \tag{14}$$

is convergent in the M^1_s-norm and

$$\|f\|_{M^1_s(\mathbb{R})} \asymp \|\{\langle f, T_{ak,bl}M_{cm,dn}\tilde{\eta}_P\rangle\}\|_{\ell^1_s(\mathbb{Z}^4)}, \quad f \in M^1_s(\mathbb{R}^2), \tag{15}$$

Furthermore, as the coefficients have decay stronger than $\ell^1(\mathbb{Z}^4)$, the convergence of the series holds in $L^1(\mathbb{R}^2)$ and in $L^2(\mathbb{R}^2)$.

Since the operator H has spreading function $\eta_H \in M^1_s(\mathbb{R}^2)$, (14) and (15) imply

$$\eta_H = \sum_{k,l,m,n\in\mathbb{Z}} c_{k,l,m,n} T_{ak,bl} M_{cm,dn} \eta_P$$

for some $\{c_{k,l,m,n}\} \in \ell^1_s(\mathbb{Z}^4)$. It is legitimate to use as identifier distributions $g \in M^\infty(\mathbb{R})$, because of the inclusion

$$\mathcal{S}(\mathbb{R}) \subset M^1_s(\mathbb{R}) \subset M^1(\mathbb{R}) \subset M^\infty(\mathbb{R}) \subset M^\infty_{-s}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

This inclusion is a consequence of [14, Proposition 11.3.4] and [14, Corollary 12.1.10]. The fact that the constant weight 1 is v_s-moderate ([14, Lemma 11.1.1]) provides the inclusion $M^1_s(\mathbb{R}) \subset M^1(\mathbb{R})$.

Next, we estimate the decay of Hg in the time and frequency domain. We shall use the fact that translation and modulation are isometries on $M^1(\mathbb{R})$ and, hence, also on $M^\infty(\mathbb{R})$ and estimate

$$\begin{aligned} |Hg(x)| &= \Big|\sum_{k,l,m,n\in\mathbb{Z}} c_{k,l,m,n} T_{ak} M_{-cm} T_{-dn} P T_{dn} M_{bl-cm}\, g(x)\Big| \\ &\leq \sum_{k,l,m,n\in\mathbb{Z}} |c_{k,l,m,n}| \cdot T_{ak-dn} |P T_{dn} M_{bl-cm}\, g(x)| \\ &\leq \|g\|_{M^\infty(\mathbb{R})} \sum_{k,l,m,n\in\mathbb{Z}} |c_{k,l,m,n}| \cdot T_{ak-dn}\phi_1(x), \end{aligned} \tag{16}$$

where $\phi_1 \in \mathcal{S}(\mathbb{R})$ is some positive function, such that $|PT_{dn}M_{bl-cm}g(x)| \leq \|g\|_{M^\infty(\mathbb{R})}\phi_1(x)$ after Lemma 4.2. For the sake of clarity we denote the expression on the right-hand side of (16) by

$$\Phi_1(x) = \sum_{k,l,m,n\in\mathbb{Z}} |c_{k,l,m,n}|\, T_{ak-dn}\phi_1(x) = \sum_{k,n\in\mathbb{Z}} \tilde{c}_{k,n} T_{ak-dn}\phi_1(x),$$

where $\tilde{c}_{k,n} = \sum_{l,m\in\mathbb{Z}} |c_{k,l,m,n}|$.

We claim that $\Phi_1(x) = O(x^{-s})$ due to $\{|c_{k,l,m,n}|\}_{k,l,m,n\in\mathbb{Z}} \in \ell^1_s(\mathbb{Z}^4) \subset \ell^1(\mathbb{Z}^4)$. Let us make a change of variables $|x|^s = y^2, y \geq 0$, which is equivalent to $x = y^{\frac{2}{s}}$, $x \geq 0$, and $x = -y^{\frac{2}{s}}$, $x < 0$. Then

$$\sup_{x\geq 0} |x^s| \cdot \Big|\sum_{k,n\in\mathbb{Z}} \tilde{c}_{k,n} T_{ak-dn}\phi_1(x)\Big| = \sup_{y\geq 0} \Big|y^2 \sum_{k,n\in\mathbb{Z}} \tilde{c}_{k,n} T_{ak-dn}\phi_1(y^{\frac{2}{s}})\Big|. \tag{17}$$

Since $y^{\frac{2}{s}}$ is monotone on $[0,\infty)$ and due to our choice $\phi_1 \in \mathcal{S}(\mathbb{R})$ (that is, ϕ_1 decays faster than the reciprocal of any polynomial on $\mathbb{R}$), $\tilde{\phi}(y) = \phi_1(y^{\frac{2}{s}})$ also decays faster than the reciprocal of any polynomial. Then we can estimate $\sup_{y\geq 0}|y^2\sum_{k,n}\tilde{c}_{k,n}T_{ak-dn}\tilde{\phi}(y)|$ by setting $\Psi_1(y) = y\tilde{\phi}(y), \Psi_2(y) = y^2\tilde{\phi}(y)$ and using the equality $y^2 = (y-ak-dn)^2-(ak-dn)^2+2y(ak-dn)$. Hence, by the triangle inequality it follows for (17) that

$$\begin{aligned} \Big|y^2\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}T_{ak-dn}\tilde{\phi}(y)\Big| \leq \Big|\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}T_{ak-dn}\Psi_2\Big| \\ +\Big|\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}(ak-dn)^2T_{ak-dn}\tilde{\phi}\Big| \\ +2\Big|\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}(ak-dn)T_{ak-dn}\Psi_1\Big|. \end{aligned} \tag{18}$$

Taking the supremum in (18) leads to

$$\begin{aligned} \sup_{y\geq 0}\Big|y^2\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}T_{ak-dn}\tilde{\phi}(y)\Big| \leq \sup_{y\geq 0}\Big|\sum_{k,n}\tilde{c}_{k,n}T_{ak-dn}\Psi_2\Big| \\ +\sup_{y\geq 0}\Big|\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}(ak-dn)^2T_{ak-dn}\tilde{\phi}\Big| \\ +2\sup_{y\geq 0}\Big|\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}(ak-dn)T_{ak-dn}\Psi_1\Big|. \end{aligned} \tag{19}$$

We compute the following upper estimate of the summands on the right-hand side in (19)

$$\begin{aligned} \sup_{y\geq 0}\Big|y^2\sum_{k,n\in\mathbb{Z}}\tilde{c}_{k,n}T_{ak-dn}\tilde{\phi}(y)\Big| \leq \sum_{k,n\in\mathbb{Z}}|\tilde{c}_{k,n}|\sup_{y\geq 0}|\Psi_2(y)| \\ +\sum_{k,n\in\mathbb{Z}}|\tilde{c}_{k,n}(ak-dn)^2|\sup_{y\geq 0}|\tilde{\phi}(y)| \\ +2\sum_{k,n\in\mathbb{Z}}|\tilde{c}_{k,n}(ak-dn)|\sup_{y\geq 0}|\Psi_1(y)|. \end{aligned} \tag{20}$$

We analyze separately the three summands from (20). Since $\tilde{\phi}, \Psi_1, \Psi_2$ belong to the Schwarz class, they are bounded and decay faster than the reciprocal of any polynomial. Also the fact that $\{c_{k,l,m,n}\}_{k,l,m,n\in\mathbb{Z}} \in \ell^1_s(\mathbb{Z}^4)$ implies the

existance of constants $C_1, C_2 > 0$ such that

$$\begin{aligned}\sum_{k,n\in\mathbb{Z}} |\tilde{c}_{k,n}| \cdot |ak-dn|^2 &\le C_1 \sum_{k,n\in\mathbb{Z}} |\tilde{c}_{k,n}|(1+a|k|+d|n|)^2 \\ &< \|\{c_{k,l,m,n}\}_{k,l,m,n\in\mathbb{Z}}\|_{\ell^1_s(\mathbb{Z}^4)} < \infty, \\ \sum_{k,n\in\mathbb{Z}} |\tilde{c}_{k,n}| \cdot |ak-dn| &\le C_2 \sum_{k,n\in\mathbb{Z}} |\tilde{c}_{k,n}|(1+a|k|+d|n|) \\ &< \|\{c_{k,l,m,n}\}_{k,l,m,n\in\mathbb{Z}}\|_{\ell^1_s(\mathbb{Z}^4)} < \infty.\end{aligned}$$

Furthermore,

$$\sum_{k,n\in\mathbb{Z}} |\tilde{c}_{k,n}| \le \|\{c_{k,l,m,n}\}_{k,l,m,n\in\mathbb{Z}}\|_{\ell^1_s(\mathbb{Z}^4)} < \infty.$$

Thus the expression on the right-hand side of (20) is bounded, implying the desired decay rate of Hg for $x > 0$. In a similar fashion we prove the decay for $x < 0$. Thus $\sup_{x\in\mathbb{R}} |x^s \Phi_1(x)| < C$ and $|Hg(x)| \le \|g\|_{M^\infty(\mathbb{R})}\varphi_1(x)$ has decay $O(x^{-s}), s > 2$.

A similar estimate can be done for the decay of the Fourier transform of Hg

$$\begin{aligned}|\mathcal{F}Hg(\omega)| &= |\sum_{k,l,m,n\in\mathbb{Z}} c_{k,l,m,n} M_{-ak} T_{-cm} M_{dn} \mathcal{F}PT_{dn} M_{bl-cm} g(\omega)| \\ &\le \sum_{k,l,m,n\in\mathbb{Z}} |c_{k,l,m,n}| \cdot T_{-cm} |\mathcal{F}PT_{dn}M_{bl-cm}g(\omega)| \\ &\le \|g\|_{M^\infty(\mathbb{R})} \sum_{k,l,m,n\in\mathbb{Z}} |c_{k,l,m,n}| \cdot T_{-cm}\phi_2(\omega),\end{aligned} \tag{21}$$

where $\phi_2 \in \mathcal{S}(\mathbb{R})$ is some positive function, such that $|\mathcal{F}PT_{dn}M_{bl-cm}g(\omega)| \le \|g\|_{M^\infty(\mathbb{R})}\phi_2(\omega)$ after Lemma 4.2. We denote the expression on the right-hand side of (21), by

$$\Phi_2(x) = \sum_{k,l,m,n} |c_{k,l,m,n}|\, T_{-cm}\phi_2(x),$$

and prove in a similar fashion that $\Phi_2(x) = O(x^{-s}), s > 2$. □

Proof of Theorem 3.6. For $m, n \in \mathbb{Z}$ and $\lambda = M(m,n)^T$, we observe that

$$\begin{aligned} H_\lambda &= T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1m+c_2n} \\ &\quad H_0 T_{d_1m+d_2n} M_{(b_1-c_1)m+(b_2-c_2)n}.\end{aligned}$$

Set $g_{m,n} = H_0 T_{d_1m+d_2n} M_{(b_1-c_1)m+(b_2-c_2)n} g$. Since $\eta_{H_0} \in M^1_s(\mathbb{R})$, $s > 2$, Lemma 4.3 implies the existence of $\phi_1(x) = O(x^{-s}), \phi_2(\omega) = O(\omega^{-s}), s > 2$ such that

$$\begin{aligned}|g_{m,n}(x)| &< \phi_1(x)\|g\|_{M^\infty(\mathbb{R})}, \\ |\mathcal{F}g_{m,n}(\omega)| &< \phi_2(\omega)\|g\|_{M^\infty(\mathbb{R})}.\end{aligned} \tag{22}$$

To prove the claim of the theorem, we show that the family

$$\{H_\lambda g\}_{\lambda\in\Lambda} = \{T_{(a_1-d_1)m+(a_2-d_2)n} M_{c_1 m + c_2 n} g_{m,n}\}_{m,n\in\mathbb{Z}^2} \subseteq L^2(\mathbb{R})$$

is a set of Gabor molecules [3, Definition 10]) localized with respect to the lattice

$$\tilde{\Lambda} = \begin{pmatrix} a_1 - d_1 & a_2 - d_2 \\ c_1 & c_2 \end{pmatrix} \mathbb{Z}^2.$$

For that purpose it suffices to show the existence of $\Psi \in W(C, \ell^2)$, such that $|\langle g_{m,n}, T_x M_\omega \gamma\rangle| < \Psi(x,\omega)$ for all $(m,n) \in \mathbb{Z}^2, (x,\omega) \in \mathbb{R}^2$. Here $\gamma(t) = e^{-t^2}$ and $W(C,\ell^2)$ denotes the Wiener amalgam space consisting of all continuous functions f on $\mathbb{R}^2$ such that the norm

$$\|f\|_{W(C,\ell^2)} = \left(\sum_{m\in\mathbb{Z}^2} \operatorname*{ess\,sup}_{z\in[0,1)^2} |f(z+m)|^2 \right)^{\frac{1}{2}} < \infty.$$

Note (22) implies

$$\begin{aligned} |\langle g_{m,n}, T_x M_\omega \gamma\rangle| &\le \langle |g_{m,n}|, T_x|\gamma|\rangle = |g_{m,n}| * \gamma(x) \\ &\le \|g\|_{M^\infty(\mathbb{R})} \cdot \phi_1 * \gamma(x)\,, \\ |\langle \mathcal{F} g_{m,n}, M_{-x} T_\omega \gamma\rangle| &\le \langle |g_{m,n}|, T_\omega|\gamma|\rangle = |g_{m,n}| * \gamma(\omega) \\ &\le \|g\|_{M^\infty(\mathbb{R})} \cdot \phi_2 * \gamma(\omega), \end{aligned}$$

see [15, 22]. Hence, by setting

$$h(t) = \|g\|_{M^\infty(\mathbb{R})} \max\{\phi_1 * \gamma(t), \phi_1 * \gamma(-t), \phi_2 * \gamma(t), \phi_2 * \gamma(-t)\},$$

we obtain

$$|\langle g_{m,n}, T_x M_\omega \gamma\rangle| \le h(\max\{|x|,|\omega|\}) = h(\|(x,\omega)\|_\infty)$$

with $|h(t)| = O(t^{-s}), s > 2$. Hence, there exists a constant c such that

$$|\langle g_{m,n}, T_x M_\omega \gamma\rangle| \le c \cdot h(\|(x,\omega)\|)$$

that can be bounded in turn pointwise by a function $\Psi(x,\omega) \in W(C,\ell^2)$. Thus, $\{H_\lambda g\}_{\lambda\in\Lambda}$ is a set of Gabor molecules.

Assume that $\overline{\operatorname{span}}\,(H_0,\Lambda)$ is identifiable by $g \in M^\infty(\mathbb{R})$. By employing [3, Theorem 8] and [2, Theorem 3] (the latter result is a Gabor molecule extension of Theorem 3.2) we obtain that the Beurling density of $\tilde{\Lambda}$, given by

$$D(\tilde{\Lambda}) = \left| \det \begin{pmatrix} a_1 - d_1 & a_2 - d_2 \\ c_1 & c_2 \end{pmatrix} \right|^{-1}$$

must be less than 1 in order for $\{H_\lambda g\}_{\lambda\in\Lambda}$ to be Riesz in $L^2(\mathbb{R})$. A computation shows that this condition follows from $\bar{D}(\Lambda) > \sqrt{2}$, yielding the bound in Theorem 3.6. $\square$

Remark 4.4 To obtain similarly a necessary density condition for $\{H_\lambda g\}_{\lambda\in\Lambda}$ to be a frame, we would have to show that $|(a_1-d_1)c_2-(a_2-d_2)c_1| = D(\tilde{\Lambda})^{-1} < 1$ follows from $\bar{D}(\Lambda) > c$ for some positive constant c. But this is not reasonable to expect, as increasing b_1 and/or b_2 greatly in

$$\begin{aligned}\bar{D}(\Lambda) = &\big[(a_1b_2-a_2b_1)^2+(a_1c_2-a_2c_1)^2\\ &+(a_1d_2-a_2d_1)^2+(b_1c_2-b_2c_1)^2\\ &+(b_1d_2-b_2d_1)^2+(c_1d_2-c_2d_1)^2\big]^{-1/2}.\end{aligned}$$

would decrease $\bar{D}(\Lambda)$.

5 Examples of identifiable operator classes, design of identifiers

To establish identifiability of $\overline{\text{span}}\,(H_0,\Lambda)$ we seek an identifier g such that any choice of coefficients $\{c_\lambda\}\in\ell^2(\Lambda)$ in $H=\sum_{\lambda\in\Lambda}c_\lambda H_\lambda$ can be computed from Hg. Equivalently, we require that $\{c_\lambda\}$ can be computed from the values of the Gabor coefficients $v_\mu=\langle Hg,\pi_1(\mu)\gamma\rangle$, $\mu\in\mathcal{M}$, where $(\gamma,\mathcal{M})$ is an L^2-Gabor frame for appropriately chosen window $\gamma\in L^2(\mathbb{R})$ and lattice $\mathcal{M}\subset\mathbb{R}^2$. Then, our task is to solve the system of linear equations

$$v_\mu=\langle Hg,\pi_1(\mu)\gamma\rangle=\sum_{\lambda\in\Lambda}c_\lambda\,\langle H_\lambda g,\pi_1(\mu)\gamma\rangle=\sum_{\lambda\in\Lambda}A_{\mu;\lambda}\;c_\lambda,\quad \mu\in\mathcal{M}$$

for $\{c_\lambda\}$. The doubly infinite matrix A has entries $A_{\mu;\lambda}=\langle H_\lambda g,\pi_1(\mu)\gamma\rangle$.

If g is such that the map $A\colon \ell^2(\Lambda)\to\ell^2(\mathcal{M})$ has a bounded left inverse then $\overline{\text{span}}\,(H_0,\Lambda)$ is identifiable.

The design of identifiers g can be carried out on the coefficient level. In fact, when $(\tilde{\gamma},\widetilde{\mathcal{M}})$ is an appropriately chosen Gabor frame for $L^2(\mathbb{R})$, or, for example, an ℓ^∞-frame for $M^\infty(\mathbb{R})$ [1], then we seek a coefficient sequence $\{d_{\widetilde{\mu}}\}$ so that the bi-infinite matrix with entries

$$A_{\mu;\lambda}=\sum_{\widetilde{\mu}\in\widetilde{\mathcal{M}}}d_{\widetilde{\mu}}\langle H_\lambda\pi_1(\widetilde{\mu})\widetilde{\gamma},\pi_1(\mu)\gamma\rangle$$

is invertible.

To illustrate the method outlined above, we give an alternative proof of one implication in Theorem 3.1 in [20]. Also see [24] for a comprehensive treatment of sampling and identification in operator Paley-Wiener spaces.

Theorem 5.1 *The operator Paley-Wiener space $OPW^2([0,a]\times[0,\frac{1}{a}])$ is identifiable.*

Proof. By definition the given operator space consists of operators $H \in HS(\mathbb{R})$ such that

$$\eta_H \in \overline{\operatorname{span}}\,\{M_{\frac{k}{a},\ell a}\chi_{[0,a]\times[0,\frac{1}{a}]},\ k,\ell \in \mathbb{Z}\},$$

that is, we consider

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & a & 0 \end{pmatrix}^T \mathbb{Z}^2.$$

Since $\{e^{2\pi i(\frac{kx}{a}+a\ell y)} : (x,y) \in [0,a]\times[0,\frac{1}{a}], k,\ell \in \mathbb{Z}\}$ forms an orthonormal basis for the space $L^2([0,a]\times[0,\frac{1}{a}])$ [19], the spreading function of any operator $H \in OPW([0,a]\times[0,\frac{1}{a}])$ has a unique expansion

$$\eta_H = \sum_{k,\ell\in\mathbb{Z}} c_{k,\ell} M_{\frac{k}{a},a\ell}\eta_0$$

with $\eta_0(x,\omega) = \chi_{[0,a]}(x)\chi_{[0,\frac{1}{a}]}(\omega)$.

Set $c_{k,\ell,m,n} = c_{k,l}\delta_{0,0}(m,n)$. We choose $\gamma = a^{-\frac{1}{2}}\chi_{[0,a]}$ and observe that the Gabor system $(\gamma, a\mathbb{Z}\times\frac{1}{a}\mathbb{Z})$ is an orthonormal basis for $L^2(\mathbb{R})$. Using the formal relationship $\langle Hg, f\rangle_{L^2(\mathbb{R})} = \langle \eta_H, V_g f\rangle_{L^2(\mathbb{R})}$ [13, Lemma 3.2] with $g = \delta_{a\mathbb{Z}} = \sum_{n\in\mathbb{Z}}\delta_{na} \in M^\infty(\mathbb{R})$[3] we compute

$$\begin{aligned} A_{p,q;k,\ell} &= \langle H_{k,\ell}\delta_{a\mathbb{Z}}, M_{\frac{p}{a}}T_{aq}\gamma\rangle \\ &= \langle T_{-aq,-\frac{p}{a}} M_{\frac{k}{a}-\frac{p}{a},a\ell}\eta_0, V_{\delta_{a\mathbb{Z}}}\gamma\rangle. \end{aligned}$$

The Zak transform Z_a satisfies the relations

$$V_{\delta_{a\mathbb{Z}}} a^{-\frac{1}{2}}\chi_{[0,a]} = Z_a a^{-\frac{1}{2}}\chi_{[0,a]}$$

and

$$a^{-\frac{1}{2}} Z_a \chi_{[0,a]}(x,\omega) = e^{2\pi i a[\frac{x}{a}]\omega},$$

for which we refer to [14, Chapter 8]. Then

$$A_{p,q;k,\ell} = \iint \chi_{[0,a]}(x+aq)e^{2\pi i\frac{(k-p)(x+aq)}{a}}\chi_{[0,\frac{1}{a}]}(\omega+\tfrac{p}{a})e^{2\pi i(\omega+\frac{p}{a})a\ell}e^{-2\pi i a[\frac{x}{a}]\omega}\,dx\,d\omega$$

We make the substitutions $y = x + aq$, $z = \omega + \frac{p}{a}$ and note that since the integrand is nonzero for $aq \le x < aq + a$, $\left[\frac{x}{a}\right] = q$, it follows that

$$\begin{aligned} A_{p,q;k,\ell} &= \int_0^a \int_0^{\frac{1}{a}} e^{2\pi i\frac{(k-p)y}{a}} e^{2\pi i a\ell z - 2\pi i a q(z-\frac{p}{a})}\,dy\,dz \\ &= e^{2\pi i(pq)} \int_0^a e^{2\pi i\frac{(k-p)y}{a}}\,dy \times \int_0^{\frac{1}{a}} e^{2\pi i a(\ell-q)z}\,dz \\ &= \delta_{p,q}(k,\ell), \end{aligned}$$

[3]Note that the inner product is still well-defined since η_H has compact support.

where we used that $\{e^{2\pi i \frac{n}{a} t} : n \in \mathbb{Z}\}$ and $\{e^{2\pi i m a t} : m \in \mathbb{Z}\}$ are orthonormal bases for $L^2[0,a]$ and $L^2[0,\frac{1}{a}]$, respectively.

The matrix $A = (A_{p,q;k,\ell})_{p,q;k,\ell\in\mathbb{Z}}$ is the identity, and thus invertible, which is what we had to prove. $\square$

Recall that $V_g f(z) = \langle f, \pi_1(z) g\rangle$, $z \in \mathbb{R}^2$ is short-time Fourier transform of $f \in L^2(\mathbb{R})$ with respect to the window $g \in L^2(\mathbb{R})$ [14, Chapter 3]. An additional positive identifiability result is the following.

Proposition 5.2 *Let $h, \{g_\lambda\}_{\lambda\in\Lambda} \in L^2(\mathbb{R})$ be such that $\{g_\lambda\}$ is a Riesz sequence for its closed linear span in L^2, $\|h\|_{L^2(\mathbb{R})} = 1$. Then the operator family $\overline{\operatorname{span}}\,\{H_\lambda, \eta_{H_\lambda} = V_h g_\lambda\}$, is identifiable.*

Proof. The Riesz sequence property of $\{g_\lambda\}$ implies that $\{\eta_{H_\lambda}\}$ is a Riesz sequence as well since it is the image of a Riesz sequence under the unitary map $V_h : g_\lambda \mapsto \eta_{H_\lambda}$ [5]. Consider the associated action of the operator H_λ on g: $H_\lambda g = g_\lambda \langle g, h\rangle$. Therefore, the entries of the matrix A have the form

$$A_{\mu;\lambda} = \sum_{\tilde{\mu}\in\tilde{\mathcal{M}}} d_{\tilde{\mu}} \langle H_\lambda \pi_1(\tilde{\mu})\tilde{\gamma}, \gamma_\mu\rangle = \sum_{\tilde{\mu}\in\tilde{\mathcal{M}}} d_{\tilde{\mu}} \langle g_\lambda, \gamma_\mu\rangle \langle \pi_1(\tilde{\mu})\tilde{\gamma}, h\rangle. \tag{23}$$

Whenever $\{\gamma_\mu\}$ is chosen biorthogonal to $\{g_\lambda\}$, (23) becomes

$$A_{\mu;\lambda} = \delta(\mu - \lambda) \sum_{\tilde{\mu}\in\tilde{\mathcal{M}}} d_{\tilde{\mu}} \langle \pi_1(\tilde{\mu})\tilde{\gamma}, h\rangle,$$

which shows that the matrix A is diagonal with non-zero entries for appropriate $\{d_{\tilde{\mu}}\}$. Hence, A is invertible. $\square$

The following examples address more involved rank-2 lattices, for simplicity of calculation, we will consider Gaussian kernels only.

Example 5.3 *Let H_0 be given by $\kappa_0(x,\omega) = e^{-\pi(x^2+\omega^2)}$, that is, $\eta_0(t,\nu) = \frac{1}{\sqrt{2}} e^{-\pi i \sqrt{2} t\nu} e^{-\frac{\pi}{2}(t^2+\nu^2)}$.*

1. *Let $\Lambda = \left(\begin{smallmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & \alpha & 0 \end{smallmatrix}\right)^T \mathbb{Z}^2$. If α, β are such that $|\alpha(\beta + \alpha\sqrt{2})| \geq \sqrt{2}, |\alpha\beta| > \sqrt{2}, |\alpha| > 1$, then the operator family $\overline{\operatorname{span}}\,(H_0, \Lambda)$ is identifiable.*
2. *Let $\Lambda = \left(\begin{smallmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta \end{smallmatrix}\right)^T \mathbb{Z}^2$. If α, β are such that $|\alpha| > 1$ and $\frac{\beta\sqrt{2}}{\alpha} \in \mathbb{Q}$, then the operator family $\overline{\operatorname{span}}\,(H_0, \Lambda)$ is identifiable.*

The pairs (α, β) satisfying the conditions in Example 5.3, Statement 1, are illustrated in Figure 1. Statement 2 of Example 5.3 indicates that identifiability may depend on conditions that are not expressible in terms of density or simple inequalities. When $\frac{\beta\sqrt{2}}{\alpha} \notin \mathbb{Q}$, the Gabor system (η_0, Λ) is not a Riesz sequence, so the problem is not considered in this paper.

The following example shows that $\bar{D}(\Lambda)$ being small does not necessarily guarantee identifiability of $\overline{\operatorname{span}}\,(H_0, \Lambda)$.

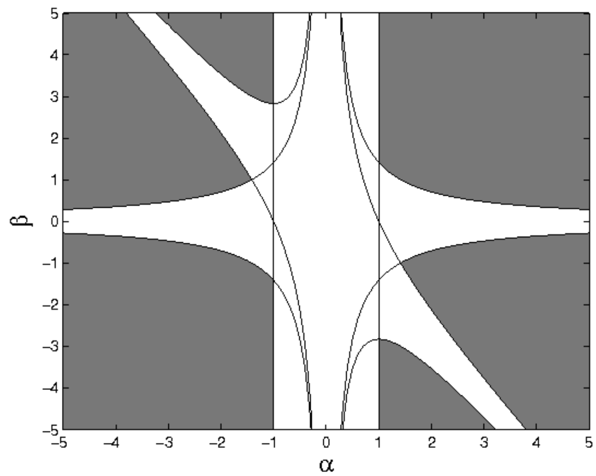

Figure 1: The set (α, β) fulfilling the conditions in Example 5.3, Statement 1, is represented by the shaded region. The region encompasses the intersection of the convex hull of the parabolas $|\alpha(\beta + \sqrt{2}\alpha)| \geq \sqrt{2}$, $|\alpha\beta| > 2$ with the subset of the plane $|\alpha| > 1$.

Example 5.4 *Let H_0 be given by $\eta_0 \in M^1_s(\mathbb{R}^2)$, $s > 2$, and let $\Lambda = \begin{pmatrix} 0 & 0 & 0 & \beta \\ \alpha & \beta & 0 & 0 \end{pmatrix}^T \mathbb{Z}^2$. If $|\alpha\beta| < 1$, then the operator family $\overline{\text{span}}\,(H_0, \Lambda)$ is not identifiable.*

The condition $|\alpha\beta| < 1$ cannot be expressed in terms of 2-Beurling density of the index set Λ, $\bar{D}(\Lambda_i) = \frac{1}{|\beta|\sqrt{\alpha^2+\beta^2}}$. In fact, for any $\epsilon > 0$, we can find α, β with $|\alpha\beta| < 1$ such that $\bar{D}(\Lambda) < \epsilon$. For instance, when $\alpha = 10^{10}, \beta = (10^{10} + 1)^{-1}$, $|\alpha\beta| < 1$, so the family $\overline{\text{span}}\,(H_0, \Lambda)$ is not identifiable, but $\bar{D}(\Lambda) \approx 10^{-20}$ is very small.

Further examples of identification using the approach described in this paper are given in [13, Sections 4.4 - 4.6].

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